

Chapter 5: Statistics for Radiation Measurement

Slide set of 120 slides based on the chapter authored by
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of the IAEA publication (ISBN 978–92–0–143810–2):
*Nuclear Medicine Physics:
A Handbook for Teachers and Students*

Objective:

To familiarize the student with the fundamental concepts of statistics for radiation measurement



IAEA

International Atomic Energy Agency

Slide set prepared in 2015
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- 5.1. Sources of error in nuclear medicine measurement
- 5.2. Characterization of data
- 5.3. Statistical models
- 5.4. Estimation of the precision of a single measurement in sample counting and imaging
- 5.5. Propagation of error
- 5.6. Applications of statistical analysis
- 5.7. Application of statistical analysis: detector performance

5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

□ Types of measurement errors:

- Blunders
- Systematic errors or accuracy of measurements
- Random errors or precision of measurements

5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

❑ Blunders

- Produce grossly inaccurate results
- Easily detected by experienced observers
- Radiation counting examples
 - Incorrect energy window setting
 - Counting heavily contaminated samples
 - Using contaminated detectors
 - High activities leading to excessive dead time effects
 - Selecting wrong patient orientation during imaging

5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

- *Systematic errors* or accuracy of measurements
 - Produce results that differ consistently from correct results by fixed amount
 - Same result may be obtained in repeated measurements, but overestimating or underestimating the true value
 - Influence accuracy of measurements
 - Results will be inaccurate or biased

5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

- ❑ *Systematic errors* or accuracy of measurements
 - Not always easy to detect → measurements may not be too different from expected results
 - Can be detected using reference standards
 - For example, use calibrated radionuclide reference standards to calibrate source calibrators to determine correction factors for each radionuclide used for patient treatment and diagnosis

5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

- *Systematic errors* or accuracy of measurements
 - Results can differ from true value by
 - constant value
 - and/or by a fraction
 - Calculate regression curve using ‘golden standard’ reference values
 - Use regression curve to convert systematic errors to more accurate value
 - **Example:** Correlate ejection fraction determined by radionuclide gated study with the ‘golden standard’ values

5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

- *Systematic errors* or accuracy of measurements
 - **Examples:**
 - Dose measured with incorrectly calibrated ionization chamber
 - Dead time losses when ^{123}I reference standard measured
 - Thyroid uptake percentage will be overestimated
 - Counting geometry different when counting samples vs reference sample

5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

❑ *Systematic errors* or accuracy of measurements

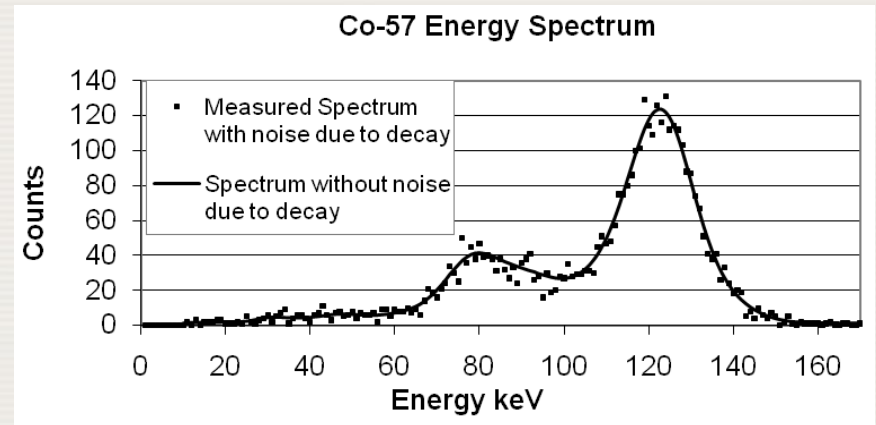
- *Examples:*

- Tracer leaks out of blood compartment in blood volume measurements
 - Will consistently overestimate the measured blood volume
- Selected background counts underestimate true ventricular background during gated blood pool studies
 - Ejection fraction will be consistently underestimated

5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

❑ *Random errors or precision of measurements*

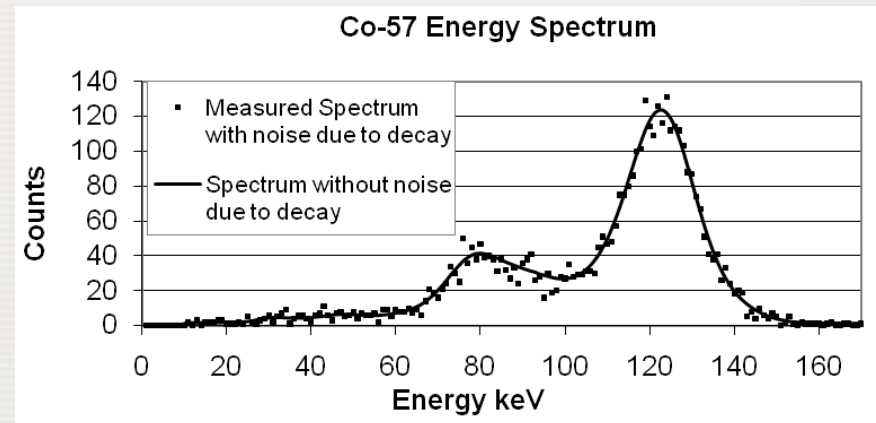
- Variations in results from one measurement to next
- Arise from:
 - Actual random variation of measured quantity
 - Physical limitations of measurement system
- Affect measurement:
 - Reproducibility
 - Precision
 - Uncertainty



5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

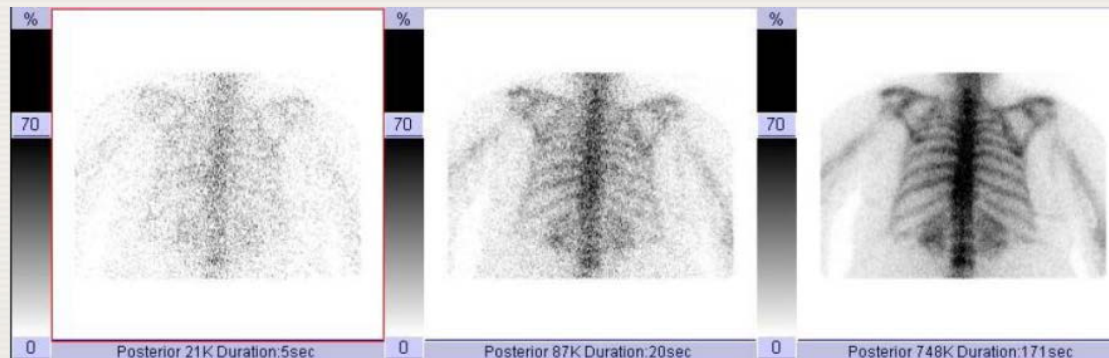
❑ *Random errors* always present in radiation measurements

- Radionuclide decay is a randomly varying quantity
- ^{57}Co energy spectrum
- Source in scattering medium
- Measured with scintillation detector
- Variation around solid line is due to random error introduced by radionuclide decay



5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

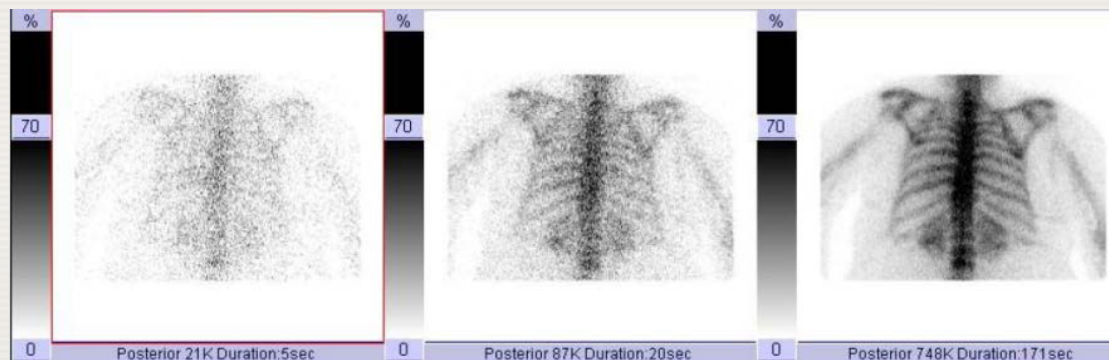
- ❑ Random errors also play a significant role in radionuclide imaging
 - Radionuclide decay errors significantly influence image visual quality
 - Number of counts (N) in each pixel is subject to random error
 - Relative random error decreases as counts per pixel increases



5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

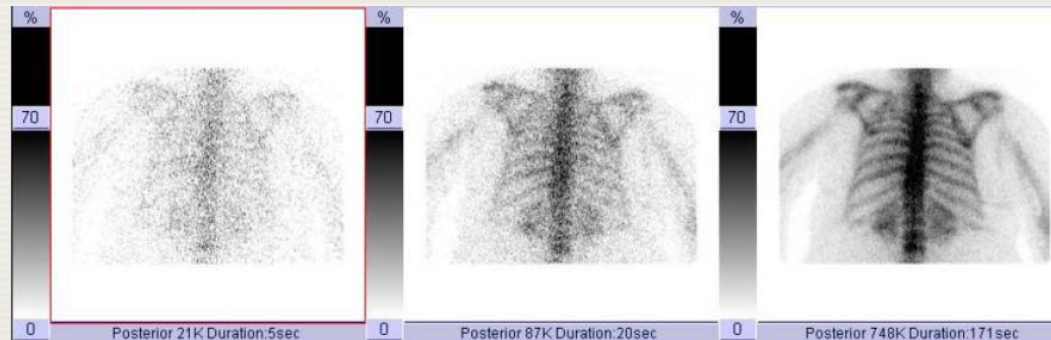
□ As N_{total} increases

- $N_{\text{per pixel}}$ increases
- Random error decreases
- Results in improved visual image quality
- Increases ability to visualize:
 - Anatomical structures
 - Tumor volumes



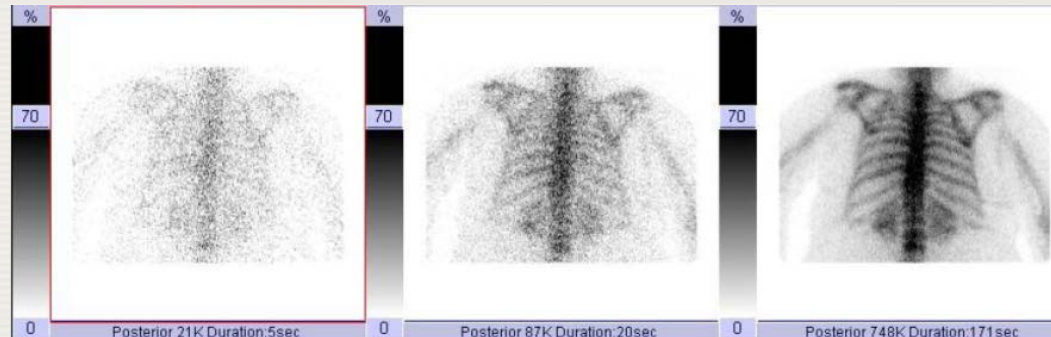
5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

- ❑ Measuring system/imaging device random error influences image quality
 - Results from energy resolution & intrinsic spatial resolution
 - Influenced by random errors during the detection of each gamma photon
 - Determines ability to reject lower energy scattered gamma photons and improve image contrast



5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

- ❑ The visual effect of the random error as a result of the measured quantity
 - ^{99m}Tc -planar bone scans (acquired on a 256×256 matrix) were acquired with a scintillation camera
 - Acquisition was terminated at 21, 87 and 748 kcnts



5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

- ❑ Measurement may be precise (small random error) but inaccurate (large systematic error), or vice versa
- ❑ **Example:** Calculation of the ejection fraction during gated cardiac studies
 - Background region of interest (ROI) selection will be exactly reproducible using software algorithm
 - Algorithm may select ROI that does not reflect true ventricular background
 - Measurement will be precise but inaccurate

5.1. SOURCES OF ERROR IN NUCLEAR MEDICINE MEASUREMENT

- ❑ Individual radiation counts of a radioactive sample may be imprecise due to random error, but average value of a number of measurements will be accurate
- ❑ It's important to analyse the random errors to determine the associated uncertainty
 - Done using methods of statistical analysis

5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.1. Data set as a list

- ❑ Measures of ***central tendency*** of data set:
 - Mean (average)
 - Median

- ❑ Measures of ***variability, random error and precision*** of data set:
 - Variance
 - Standard deviation
 - Fractional standard deviation

- ❑ ***Examples:***
 - Data set obtained from long lived radioactive sample

5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.1. Data set as a list

□ *Examples:*

- Data set obtained from long lived radioactive sample
- Counted repeatedly
 - All done under same conditions
 - Properly operating counting system
- Sample disintegration rate has random variations from one moment to next
- Number of counts recorded in successive measurements is not the same
 - Due to random errors

5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.1. Data set as a list

- Calculation for experimental mean of a data set:
 - N independent measurements of same physical quantity
 - $x_1, x_2, x_3, \dots, x_i, \dots, x_N$
 - The mean given by:

$$\begin{aligned}\bar{x}_e &= \frac{x_1 + x_2 + \dots + x_N}{N} \\ &= \frac{\sum_{i=1}^N x_i}{N}\end{aligned}$$

5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.1. Data set as a list

□ Procedure for obtaining median

- Sort list of measurements by size
- Median:
 - Middlemost measurement if N is odd
 - Average of 2 middlemost measurements if N is even
 - Example
 - $N=5$
 - Measurements: 7, 13, 6, 10 and 14
 - Sorted by size: 6, 7, 10, 13 and 14
 - Median = 10
- Less affected by outliers than the mean
- Outlier is a blunder
 - Much greater or much less than the others.

5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.1. Data set as a list

□ **Variance:**

$$\sigma_e^2 = \frac{(x_1 - \bar{x}_e)^2 + (x_2 - \bar{x}_e)^2 + \dots + (x_N - \bar{x}_e)^2}{N - 1}$$
$$= \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x}_e)^2$$

□ **Standard Deviation:**

$$\sigma_e = \sqrt{\sigma_e^2}$$

□ **Fractional standard deviation, σ_{eF}** (fractional error or coefficient of variation) :

$$\sigma_{eF} = \frac{\sigma_e}{\bar{x}_e}$$

- Inverse of σ_{eF} referred to as the signal to noise ratio.

5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.2. Data set as a relative frequency distribution function

□ Often convenient to represent data set by *relative frequency distribution function* $F(x)$

- $F(x)$ = measured value relative frequency

$$F(x) = \frac{\text{Number of occurrences of the value } x \text{ in each bin}}{N}$$

- $F(x)$ is normalized:

$$\sum_{x=0}^{\infty} F(x) = 1$$

- Represents all information in original data set in list format
- Useful quick visual summary of measurement values distribution
- Can be used to identify outliers

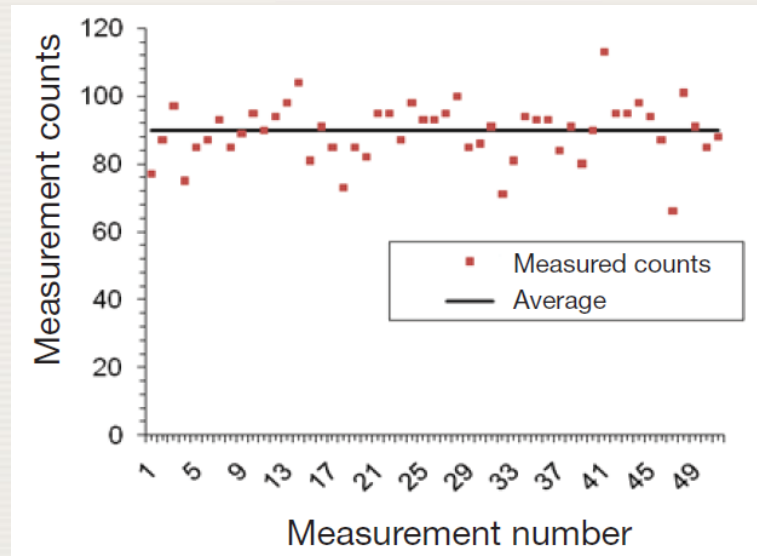
5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.2. Data set as a relative frequency distribution function

□ Example of relative frequency distribution application

- Scintillation counter measurements appear noisy due to random error
- Measurements fluctuate randomly around mean = 90 counts.



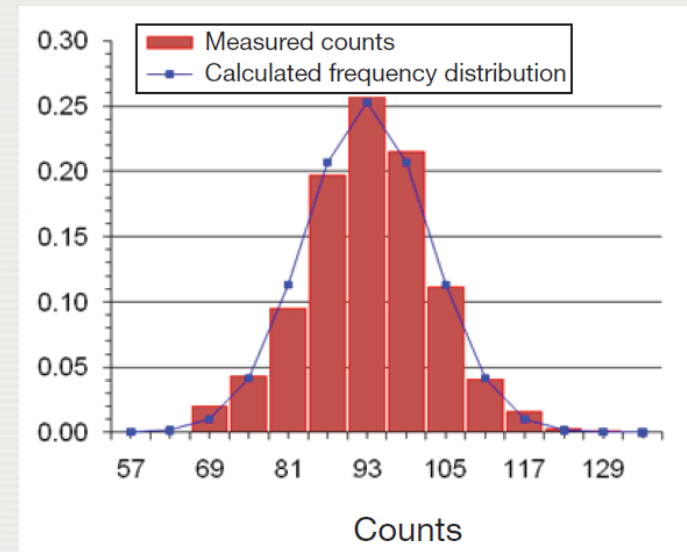
5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.2. Data set as a relative frequency distribution function

□ Histogram (red bars) of $F(x)$ of measurements fluctuations

- Plot shows relative frequency of measured counts
- X-axis = range of possible counts
 - At 6 count intervals
- Y-axis = relative frequencies of particular count values occur
- Most common value = mean
 - e.g. 26% of measurements in plot are near the mean = 90 counts
- Blue curve: expected calculated normal distribution



5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.2. Data set as a relative frequency distribution function

□ Measures of central tendency using $F(x)$:

- Mean (average)
- Median
 - Value at which integral of $F(x)$ is 0.5
 - Half of measurements will be $<$ and half will be $>$ median
- Mode
 - Most frequent value OR
 - Value at maximum probability of $F(x)$

5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.2. Data set as a relative frequency distribution function

□ Mean of $F(x)$:

$$\begin{aligned}\bar{x}_e &= \frac{\sum_{i=1}^N x_i}{N} \\ &= \sum_{i=1}^N \frac{[\text{value of bin } (i)] [\text{number of occurrences of the value } x_i]}{N} \\ &= \sum_{i=1}^N x_i F(x_i)\end{aligned}$$

□ Experimental sample variance using $F(x)$:

$$\sigma_e^2 = \frac{N}{N-1} \sum_{i=1}^N (x_i - \bar{x}_e)^2 F(x_i)$$

5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.2. Data set as a relative frequency distribution function

- $F(x)$ provides insight on the precision of the experimental sample mean and of a single measurement
- The value of the true mean is not usually known but the experimental sample mean can be used as an estimate of the true mean

$$\left(\bar{x}_t\right) \approx \left(\bar{x}_e\right)$$

5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.2. Data set as a relative frequency distribution function

- ❑ Often impractical to obtain multiple measurements, especially for radionuclide imaging & nuclear measurements on patients
- ❑ $F(x)$ will determine precision of single measurement as an estimate of true value
- ❑ Probability that single measurement will be close to \bar{x}_t depends on the *dispersion* (relative width) $F(x)$. This is expressed by the σ^2 or σ
- ❑ The *standard deviation* σ is a number such that **68.3%** of the measurement results fall within $\pm\sigma$ of \bar{x}_t
 - I.e. If x = result of a given measurement, there is a 68.3% chance that is within the range $x \pm \sigma$
 - Called the 68.3% confidence interval for \bar{x}_t

5.2. CHARACTERIZATION OF DATA

5.2.1. Measures of central tendency and variability

5.2.1.2. Data set as a relative frequency distribution function

□ CONFIDENCE LEVELS IN RADIATION MEASUREMENTS

Range	Confidence level for true mean \bar{x}_t
$\bar{x}_t \pm 0.675\sigma$	50.0
$\bar{x}_t \pm 1.000\sigma$	68.3
$\bar{x}_t \pm 1.640\sigma$	90.0
$\bar{x}_t \pm 2.000\sigma$	95.0
$\bar{x}_t \pm 3.000\sigma$	99.7

5.3. STATISTICAL MODELS

- $F(x)$ of many repetitions can be predicted under certain conditions
- A *measurement* is defined as counting the number of **successes** x resulting from a given **number of trials** n .
- Assumptions:
 - Only two results are possible for each trial (i.e. binary process): either a success or not
 - Probability of success p is constant for all trials.
 - e.g.

Trial	Definition of success	Probability of success p
Tossing a coin	Heads	1/2
Rolling a die	A six	1/6
Observing a given radionuclide for time t	The nucleus decays during observation	$(1 - e^{-\lambda t})$
Observing a given γ ray over a distance x in an attenuating medium	The γ ray interacts with the medium during observation	$(1 - e^{-\mu x})$
Observing light photons generated in a scintillator	An electron is ejected from the photocathode	1/5

5.3. STATISTICAL MODELS

- ❑ The third example gives the basis for counting nuclear radiation events
- ❑ A trial consists of observing a given radioactive nucleus for a period of time t
- ❑ The number of trials $n =$ number of nuclei in the measured sample
- ❑ Measurement consists of counting nuclei that undergo decay
- ❑ The probability of success, p , for radioactive decay:

$$p = (1 - e^{-\lambda t})$$

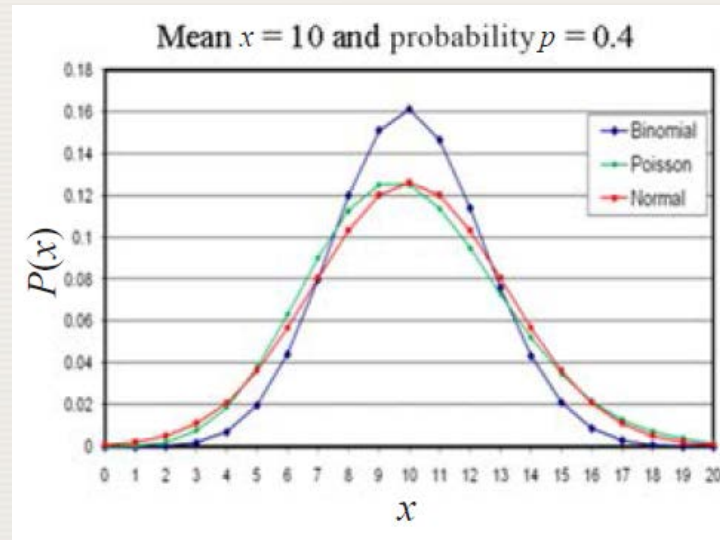
- $\lambda =$ radionuclide decay constant

5.3. STATISTICAL MODELS

5.3.1. When binomial, Poisson & normal distributions are applicable

5.3.1.1. Binomial distribution

- ❑ This is the most general model and is widely applicable to all constant p processes.
- ❑ Rarely used in nuclear decay applications.
- ❑ Must be used is when a radionuclide with a very short half-life is counted with a high counting efficiency.



5.3. STATISTICAL MODELS

5.3.1. When binomial, Poisson & normal distributions are applicable

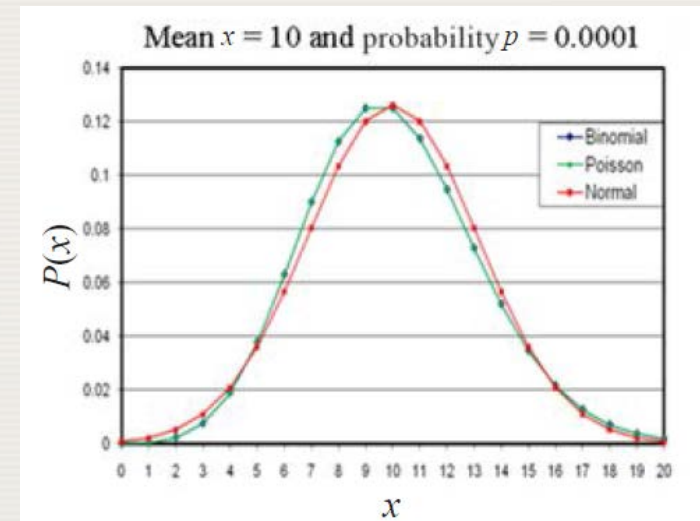
5.3.1.2. Poisson distribution

□ Poisson distribution

- Direct mathematical simplification when *probability of success p is small*
- Must be used because experimental distribution is asymmetric but normal distribution is always symmetrical

□ For nuclear counting, used when:

- Observation time $\ll T_{1/2}$ of source
- Detection efficiency is low



5.3. STATISTICAL MODELS

5.3.1. When binomial, Poisson & normal distributions are applicable

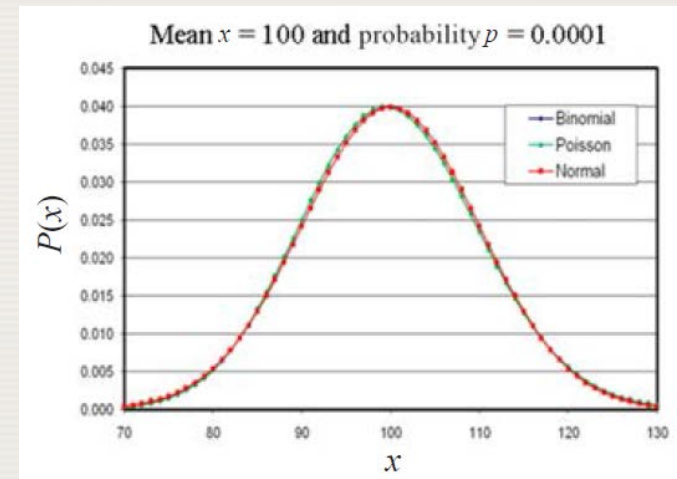
5.3.1.3. Gaussian or normal distribution

□ Gaussian (normal) distribution

- Further simplification of binomial if the mean number of successes is relatively large (>30)
- Experimental distribution will be symmetrical

□ All 3 model distributions becomes identical when:

- p is small
- Large enough number of trials such that the expected mean number of successes is large.



5.3. STATISTICAL MODELS

5.3.2. Binomial distribution

□ Most general of the statistical models discussed

- n = number of trials, is *always an integer*
- Each trial has p = success probability
- $P(x)$ = predicted probability of counting exactly x successes :

$$P(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

- Properties

- Must be normalized

$$\sum_{x=0}^n P(x) = 1$$

- Mean value, \bar{x} , using the definition of mean and of $P(x)$:

$$\bar{x} = \sum_{x=0}^n xP(x) = pn$$

5.3. STATISTICAL MODELS

5.3.2. Binomial distribution

□ The predicted variance σ^2 is given by:

$$\sigma^2 = \sum_{x=0}^{\infty} (x - \bar{x})^2 P(x)$$

- Substituting for $P(x)$ and np , \bar{x} :

$$\sigma^2 = np(1-p) = \bar{x}(1-p)$$

□ The standard deviation is:

$$\sigma = \sqrt{np(1-p)} = \sqrt{\bar{x}(1-p)}$$

5.3. STATISTICAL MODELS

5.3.2. Binomial distribution

- The fractional standard deviation σ_F is given by:

$$\sigma_F = \frac{\sqrt{np(1-p)}}{np} = \sqrt{\frac{(1-p)}{np}}$$

$$\sigma_F = \frac{\sqrt{\bar{x}(1-p)}}{\bar{x}} = \sqrt{\frac{(1-p)}{\bar{x}}}$$

- σ predicts amount of fluctuation inherent in binomial distribution in terms of n & p

5.3. STATISTICAL MODELS

5.3.2. Binomial distribution

5.3.2.1. Application example of binomial distribution

□ The operation of a scintillation detector consists of:

- Scintillation crystal mounted on photomultiplier (PMT) in a light tight construction
- Photon interacts with crystal & generates n light γ 's
- Light γ 's eject x electrons (e^-) from PMT photocathode
- Electrons multiplied to form pulse that can be further processed
- n , x & *multiplication* factor vary statistically during γ detection
- This variation determines the energy resolution of the system

5.3. STATISTICAL MODELS

5.3.2. Binomial distribution

5.3.2.1. Application example of binomial distribution

□ **Example:** ejection of e^- 's from photocathode

- Typical scintillation counter values for 142 keV ^{99m}Tc γ 's:
 - Need: 100eV per scintillation γ
 - $n = 142\,000 \text{ eV}/100\text{eV}/\text{photon} = 1420$ light photons emitted
 - x electrons generate these fall on PMT photocathode per gamma ray absorbed.
 - 5 light γ 's required to eject one e^-
 - Binomial distribution probability of light γ ejecting an e^- is:

$$p = 1/5$$

5.3. STATISTICAL MODELS

5.3.2. Binomial distribution

5.3.2.1. Application example of binomial distribution

- Predicted mean number of e⁻s ejected per gamma ray:

$$\bar{x} = pn = \frac{1}{5} \times 1420 = 284 \text{ e}^-$$

- σ & σ_F for x (no. of e⁻s ejected) calculated using binomial distribution:

$$\sigma = \sqrt{\bar{x}(1-p)} = \sqrt{284 \left(1 - \frac{1}{5}\right)} = 15$$

$$\sigma_F = \sqrt{\frac{(1-p)}{\bar{x}}} = \sqrt{\frac{(1-1/5)}{284}} = 0.053$$

5.3. STATISTICAL MODELS

5.3.2. Binomial distribution

5.3.2.1. Application example of binomial distribution

- The electron ejection stage at the photocathode contributes 5.3% to the overall σ
- The variation in x will influence the pulse height obtained for each gamma ray.
- The variation in the pulse height during the detection of gamma rays will determine the width of the photopeak and the energy resolution of the system

5.3. STATISTICAL MODELS

5.3.3. Poisson distribution

- Examples of binary processes characterized by low probability of success for each individual trial:
 - Nuclear counting & imaging applications with
 - large numbers of radionuclides in sample
 - Nuclear counting & imaging applications with large number of trials
 - Relatively small fraction gives rise to recorded counts.
 - Administered imaging radionuclide:
 - For every gamma that interacts with tissue many are emitted
 - Many gammas strike the detector for every single recorded interaction.
 - Approximation of small p holds under these conditions
 - Some mathematical simplifications can be applied to the binomial distribution.

5.3. STATISTICAL MODELS

5.3.3. Poisson distribution

□ Poisson distribution

- The binomial distribution reduces to the form:

$$P(x) = \frac{(pn)^x e^{-pn}}{x!}$$

- $pn = \bar{x}$ holds for this distribution

- Final form:
$$P(x) = \frac{(\bar{x})^x e^{-\bar{x}}}{x!}$$

- Only one parameter to get $P(x)$: \bar{x}
 - Useful when:
 - \bar{x} can be measured/estimated
 - there is no information about p or n
 - e.g. nuclear counting and imaging

5.3. STATISTICAL MODELS

5.3.3. Poisson distribution

□ Poisson distribution properties

- Normalized frequency distribution
- Mean value (or first moment) same as that of the binomial distribution:

$$\bar{x} = \sum_{x=0}^{\infty} xP(x) = pn$$

- Predicted variance differs from that of the binomial distribution :

$$\sigma^2 = \sum_{x=0}^{\infty} (x - \bar{x})^2 P(x) = pn = \bar{x}$$

- Predicted standard deviation & fractional standard deviation

σ_F :

$$\sigma = \sqrt{\sigma^2} = \sqrt{\bar{x}} \qquad \sigma_F = \frac{\sigma}{\bar{x}} = \frac{1}{\sqrt{\bar{x}}} = \frac{1}{\sigma}$$

5.3. STATISTICAL MODELS

5.3.4. Normal distribution

□ Normal distribution

- $p < 1$ & $\bar{x} > 30$
- The binomial distribution reduces to the form:

$$P(x) = \frac{1}{\sqrt{2\pi\bar{x}}} e^{-\left(\frac{(x-\bar{x})^2}{2\bar{x}}\right)}$$

- Only defined for integer values of x
- Always symmetrical or 'bell-shaped'
- Properties shared with the Poisson distribution:
 - It is normalized
 - Characterized by a single parameter
 - Predicted variance: $\sigma^2 = \bar{x}$
 - Predicted standard : $\sigma = \sqrt{\bar{x}}$
 - Predicted fractional standard deviation: $\sigma_F = 1/\sigma$

5.3. STATISTICAL MODELS

5.3.4. Normal distribution

5.3.4.1. Continuous normal distribution: confidence intervals

- ❑ Number of possible sample outcomes increases with sample size
- ❑ As the sample size approaches infinity the distribution becomes continuous
- ❑ Some random variables are essentially continuous and have continuous distributions (e.g. such as height and weight)
- ❑ In these situations, p is not small as was assumed for the discrete Poisson and normal distributions, and the equation does not apply
- ❑ The continuous normal distribution is given by:

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2}$$

5.3. STATISTICAL MODELS

5.3.4. Normal distribution

5.3.4.1. Continuous normal distribution: confidence intervals

□ Continuous normal distribution properties:

- Continuous
- Symmetrical
- Both tails extend to infinity
- Mean = median = mode
- Described by 2 parameters:
 - mean \bar{x}
 - Determines location curve centre
 - standard deviation σ
 - Represents the spread around \bar{x}

5.3. STATISTICAL MODELS

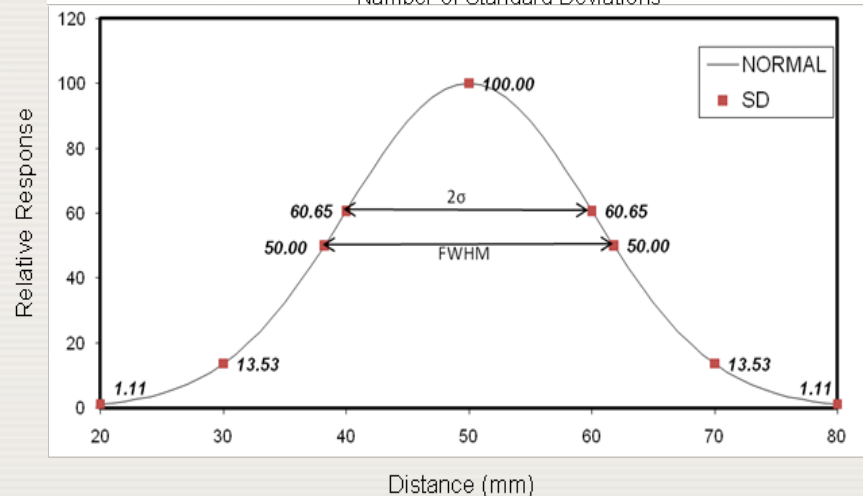
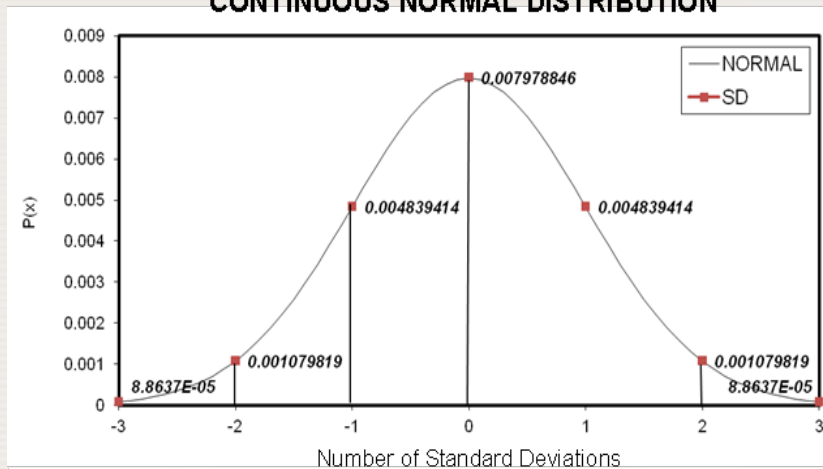
5.3.4. Normal distribution

5.3.4.1. Continuous normal distribution: confidence intervals

Of the total curve area:

- 68% between $\bar{x} \pm \sigma$
- 95% between $\bar{x} \pm 2\sigma$
- 99.7% between $\bar{x} \pm 3\sigma$

CONTINUOUS NORMAL DISTRIBUTION



5.3. STATISTICAL MODELS

5.3.4. Normal distribution

5.3.4.2. Continuous normal distribution: applications in medical physics

- Normal distribution often used to fit experimental data
- The equation is modified so that the maximum value of the distribution at \bar{x} is normalized to 100:

$$P(x) = 100 e^{-\frac{1}{2} \left(\frac{x - \bar{x}}{\sigma} \right)^2}$$

5.3. STATISTICAL MODELS

5.3.4. Normal distribution

5.3.4.2. Continuous normal distribution: applications in medical physics

- The spatial resolution of imaging devices is determined as the full width at half maximum (FWHM) of a normal distribution fitted to point or line spread function

- Relationship between FWHM & σ for a normal distribution:
 - setting $P(x) = 50$ and solving

$$\text{FWHM} = 2.355\sigma$$

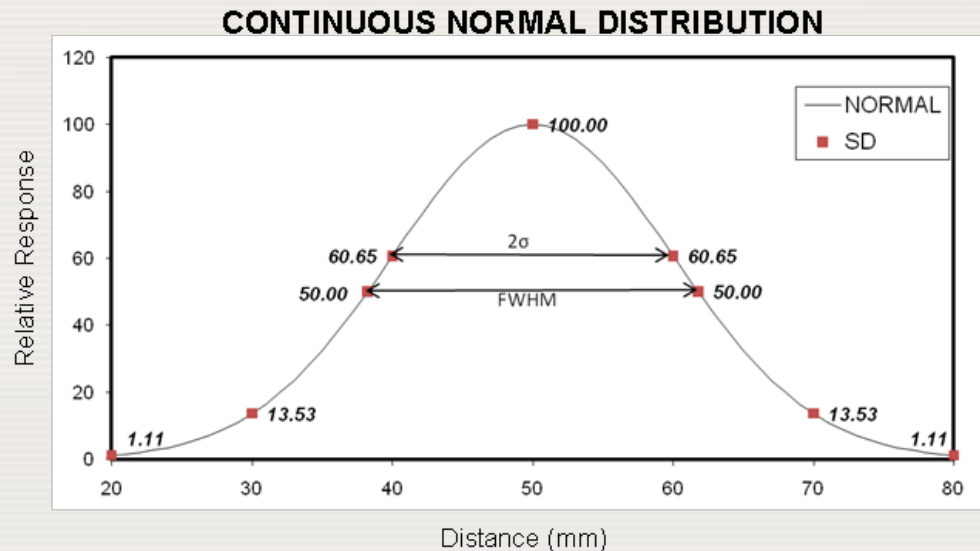
5.3. STATISTICAL MODELS

5.3.4. Normal distribution

5.3.4.2. Continuous normal distribution: applications in medical physics

□ *Example:*

- Line source response for a scintillation camera
- Fitted to a normal distribution
- $\sigma = 10\text{mm}$
- Image resolution = FWHM = 23.6 mm



5.3. STATISTICAL MODELS

5.3.4. Normal distribution

5.3.4.2. Continuous normal distribution: applications in medical physics

- ❑ The photopeak in nuclear spectroscopy can be fitted to a normal distribution
- ❑ Fractional energy resolution, R_E , of scintillation detectors:

$$R_E = \frac{\text{FWHM}}{E} = \frac{2.355\sigma}{E}$$

- ❑ The energy spectrum in medical physics applications is measured in kiloelectronvolts (keV) or megaelectronvolts (MeV)

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT IN SAMPLE COUNTING AND IMAGING

5.4.1. Assumption

- ❑ Counting statistics is useful when a single measurement of a quantity is available & the uncertainty associated with that measurement is required

- ❑ The square root of the sample variance, σ
 - Measure of deviation of any one measurement from the true mean
 - Serves as an index of degree of precision associated with a measurement from that set

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT IN SAMPLE COUNTING AND IMAGING

5.4.1. Assumption

- For a single measurement, x , is available
 - Sample variance cannot be calculated directly
 - σ must be estimated by analogy with appropriate statistical model
 - Appropriate theoretical distribution matched to data if x is drawn from a population whose distribution is predicted by Poisson or Gaussian
 - *Assume the distribution mean = x*

$$\bar{x} \approx x$$

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT IN SAMPLE COUNTING AND IMAGING

5.4.1. Assumption

□ Assuming $\bar{x} \approx x$, $P(x)$ is defined for all values of x

□ Expected sample variance, s^2 , expressed in terms of σ :

$$s^2 = \sigma^2 = \bar{x} \approx x$$

□ The best estimate of σ from \bar{x} , which should typify a single measurement x , is given by:

$$\sqrt{s^2} = \sigma \approx \sqrt{x}$$

□ Assuming $P(x)$ is Gaussian with large x

□ The range of values $x \pm \sigma$ (or $x \pm \sqrt{x}$) will contain the true mean with a 68% probability

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT IN SAMPLE COUNTING AND IMAGING

5.4.1. Assumption

- If $x=100$, then $\sigma \sqrt{N} x = \sqrt{100} = 10$
- Conventional choice for quoting uncertainty of single measurement x :
 - $x \pm \sigma$
 - Interval is expected to contain true mean with probability of 68%
 - Probability can be increased by expanding interval
 - **Example:**
 - for $P(x)=99\%$, interval must be expanded by 2.58σ
 - In the example, the range is 100 ± 25.8
- The associated probability level should be stated under methods when errors are reported

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT IN SAMPLE COUNTING AND IMAGING

5.4.1. Assumption

- Options available in quoting uncertainty to be associated with a single measurement are shown

Interval (relative σ)	Interval (values)	Probability that the true mean \bar{x} is included (%)
$x \pm 0.67\sigma$	93.3–106.7	50
$x \pm 1.00\sigma$	90.0–110.0	68
$x \pm 1.64\sigma$	83.6–116.4	90
$x \pm 2.00\sigma$	80.0–120.0	95
$x \pm 2.58\sigma$	74.2–125.8	99
$x \pm 3.00\sigma$	70.0–130.0	99.7

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT

5.4.2. The importance of the fractional σ_F

- The precision, σ , will increase proportionally to \sqrt{x}
 - i.e. if x increases, σ will also increase
 - Increase $\sigma <$ increase in x

- The relation σ & x is best demonstrated by calculating σ_F :

$$\sigma_F = \frac{\sigma}{x} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

- The number of counts or single measurement value x completely determines σ_F
- σ_F decreases as x increases
- Minimum number of counts must be accumulated to achieve a given σ_F

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT

5.4.2. The importance of the fractional σ_F

□ *Example :*

- If 100 counts recorded $\rightarrow \sigma_F = 10\%$
- If 10 000 counts recorded $\rightarrow \sigma_F = 1\%$
- Demonstrates the importance of acquiring enough counts to meet the required precision

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT

5.4.2. The importance of the fractional σ_F

- ❑ Easier to achieve σ_F when measuring samples in counting tubes than in vivo on patients
- ❑ Single measurement from high activity sample will be obtained in a short time
- ❑ To achieve same σ_F using low activity sample measurement time will have to be increased
- ❑ Can be obtained by using automatic counters set to stop counting after a preset time or preset counts have been reached

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT

5.4.2. The importance of the fractional σ_F

- Single measurement precision is very important during imaging
 - Low precision obtained if number of counts (N) acquired in a picture element or pixel is low
 - There will then be a wide range of fluctuations between adjacent pixels
 - ✓ Poor image quality
 - ✓ Only possible to identify: large defect volumes or defects with a high contrast

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT

5.4.2. The importance of the fractional σ_F

- ❑ Detection of a defect requires that N from defect lie outside the background range: $x_b \pm 2\sigma_b$
- ❑ N in a target volume will be determined by acquisition time, activity in target volume and the sensitivity of the measuring equipment

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT

5.4.2. The importance of the fractional σ_F

- ❑ Imaging equipment sensitivity can be increased by increasing spatial resolution in terms of FWHM

- ❑ To obtain images with the max diagnostic value during visual interpretation there is a trade off between:
 - Single sample counting precision
 - Spatial resolution of imaging device

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT

5.4.2. The importance of the fractional σ_F

- Counting statistics are very important during image quantification
 - Accumulated counts within a target volume (e.g. an organ) have to be accurately determined
 - Accurate quantification requires corrections be made for:
 - Background activity
 - Attenuation
 - Scatter contributions
 - ***These procedures further reduce the precision of quantification***

5.4. ESTIMATION OF THE PRECISION OF A SINGLE MEASUREMENT

5.4.3. Caution on the use of the estimate of the precision of a single measurement in sample counting and imaging

- ❑ All conclusions based on measured number of success
- ❑ Nuclear measurements/imaging
 - Single measurement precision using $\sigma = \sqrt{x}$ can only be applied if x represents number of events recorded in a given observation time
 - For example does not apply to:
 - ✓ Counting rates
 - ✓ Sums or differences of counts
 - ✓ Averages of independent counts
 - ✓ Pixel counts following tomographic image reconstruction
 - ✓ Any derived quantity
- ❑ Quantity calculated as function of number of recorded counts
- ❑ Error must be calculated according to error propagation methods outlined

5.5. PROPAGATION OF ERROR

- ❑ Preceding section described methods for estimating random error or precision of single measurement

- ❑ Most diagnostic nuclear medicine procedures involve multiple measurements/imaging procedures for calculation of results
 - Examples:
 - thyroid iodine uptake
 - ejection fraction
 - renal clearance
 - blood volume
 - red cell survival time

- ❑ Internal dosimetry is performed using nuclear measurements and imaging data

5.5. PROPAGATION OF ERROR

- ❑ Corresponding precision in derived quantity is estimated by propagating initial measurement errors through the calculations performed to arrive at the result
- ❑ The variables used in the calculation of errors must be independent to avoid correlation effects
- ❑ Assumption:
 - Nuclear measurement error arises **only** from random fluctuations in decay rate
 - Statistically independent of other errors

5.5. PROPAGATION OF ERROR

- Error propagation formulas apply to measurements obtained from
 - Continuous distribution
 - Represented by $x_1, x_2, x_3...$ with variances $\sigma(x_1)^2, \sigma(x_2)^2, \sigma(x_3)^2...$
 - e.g. to estimate precision in height and weight measurements
 - Poisson distributions
 - Represented by $N_1, N_2, N_3...$ with variances $\sigma(N_1)^2, \sigma(N_2)^2, \sigma(N_3)^2...$
 - Discrete normal distributions.
 - Represented by $N_1, N_2, N_3...$ with variances $\sigma(N_1)^2, \sigma(N_2)^2, \sigma(N_3)^2...$

5.5. PROPAGATION OF ERROR

5.5.1. Sums and differences

□ If $x_s = x_1 \pm x_2 \pm x_3 \dots$

- Variance :

$$\sigma(x_1 \pm x_2 \pm x_3 \dots)^2 = \sigma(x_1)^2 + \sigma(x_2)^2 + \sigma(x_3)^2 \dots$$

- Standard deviation :

$$\sigma(x_1 \pm x_2 \pm x_3 \dots) = \sqrt{\sigma(x_1)^2 + \sigma(x_2)^2 + \sigma(x_3)^2 \dots}$$

- Fractional standard deviation:

$$\sigma_F(x_1 \pm x_2 \pm x_3 \dots) = \frac{\sqrt{\sigma(x_1)^2 + \sigma(x_2)^2 + \sigma(x_3)^2 \dots}}{x_1 \pm x_2 \pm x_3 \dots}$$

5.5. PROPAGATION OF ERROR

5.5.1. Sums and differences

□ For measurements with Poisson or discrete normal distribution:

- Variance :

$$\sigma(N_1 \pm N_2 \pm N_3 \dots)^2 = N_1 \pm N_2 \pm N_3 \dots$$

- Standard deviation:

$$\sigma(N_1 \pm N_2 \pm N_3 \dots) = \sqrt{N_1 + N_2 + N_3 \dots}$$

- Fractional standard deviation:

$$\sigma_F(N_1 \pm N_2 \pm N_3 \dots) = \frac{\sqrt{N_1 \pm N_2 \pm N_3 \dots}}{N_1 \pm N_2 \pm N_3 \dots}$$

- ***Equations apply to mixed combinations of sums and differences***

5.5. PROPAGATION OF ERROR

5.5.1. Sums and differences

- The influence on σ & σ_F of *adding/subtracting*:

	$N_2 \ll N_1$			$N_2 \approx N_1$		
	N	σ	σ_F	N	σ	σ_F
N_1	500	22.4	0.0447	500	22.4	0.0447
N_2	10	3.2	0.3162	450	21.2	0.0471
$N_1 - N_2$	490	22.6	0.0461	50	30.8	0.6164
$N_1 + N_2$	510	22.6	0.0443	950	30.8	0.0324

5.5. PROPAGATION OF ERROR

5.5.1. Sums and differences

□ Conclusions:

- σ is the same for $N_1 - N_2$ & $N_1 + N_2$ for the same values of N_1 & N_2
- σ_F for differences is large when the differences between the values are small

➡ Background must be limited to as low a value as possible

- Imaging scatter/background corrections involve subtraction

➡ The statistical information deteriorates as a result of the increased uncertainty in the pixel values

5.5. PROPAGATION OF ERROR

5.5.2. Multiplication and division by a constant

□ Multiplication:

- $x_M = Ax$, where A is a constant

$$\sigma_M = A\sigma_x$$

$$\sigma_F = \frac{A\sigma_x}{Ax} = \frac{\sigma_x}{x}$$

□ Counting measurements / Poisson / discrete normal distribution:

- $x_M = AN$

$$\sigma_M = A\sqrt{N}$$

$$\sigma_F = \frac{1}{\sqrt{N}}$$

5.5. PROPAGATION OF ERROR

5.5.2. Multiplication and division by a constant

□ Division:

- $x_D = x/B$, where B is a constant.

$$\sigma_M = \frac{\sigma_x}{B}$$

$$\sigma_F = \frac{\sigma_x}{B} \frac{B}{x} = \frac{\sigma_x}{x}$$

□ Counting measurements / Poisson / discrete normal distribution:

- $x_D = N/B$

$$\sigma_M = \frac{\sqrt{N}}{B}$$

$$\sigma_F = \frac{1}{\sqrt{N}}$$

5.5. PROPAGATION OF ERROR

5.5.3. Products and ratios

- The uncertainty in the product or ratio of a series of measurements $x_1, x_2, x_3 \dots$ is expressed in terms of the fractional uncertainties in the individual results, $\sigma_F(x_1), \sigma_F(x_2), \sigma_F(x_3) \dots$

- x_P of measurements with continuous normal distribution:

$$x_P = x_1 \times x_2 \times x_3 \dots \quad x_P = x_1 \div x_2 \div x_3 \dots$$

- Fractional variance

$$\sigma_F(x_P)^2 = \sigma_F(x_1)^2 + \sigma_F(x_2)^2 + \sigma_F(x_3)^2 \dots$$

- Fractional standard deviation

$$\sigma_F(x_P) = \sqrt{\sigma_F(x_1)^2 + \sigma_F(x_2)^2 + \sigma_F(x_3)^2 \dots}$$

- Standard deviation

$$\sigma(x_P) = \sqrt{\sigma_F(x_1)^2 + \sigma_F(x_2)^2 + \sigma_F(x_3)^2 \dots} \times x_P$$

5.5. PROPAGATION OF ERROR

5.5.3. Products and ratios

□ For counting / Poisson / discrete normal distribution measurements:

- x_p of measurements with continuous normal distribution:

$$N_P = N_1 \times N_2 \times N_3 \dots$$

$$N_P = N_1 \div N_2 \div N_3 \dots$$

- Fractional variance

$$\sigma_F(N_P)^2 = \frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} \dots$$

- Fractional standard deviation

$$\sigma_F(N_P) = \sqrt{\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} \dots}$$

- Standard deviation

$$\sigma(N_P) = \sqrt{\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} \dots} \times N_P$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.1. Multiple independent counts

5.6.1.1. Sum of multiple independent counts

□ If there are

- n repeated counts from the same source
- for equal counting times
- the results of the measurements are $N_1, N_2, N_3, \dots, N_n$ and their sum is N_s , then:

$$N_s = N_1 + N_2 + N_3 \dots$$

$$\sigma_{N_s} = \sqrt{N_1 + N_2 + N_3 \dots} = \sqrt{N_s}$$

□ σ for the sum of all counts is the same as if the measurement had been carried out by performing a single count, extending over the period represented by all of the counts.

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.1. Multiple independent counts

5.6.1.1. Sum of multiple independent counts

- The mean value of the n independent counts:

$$\bar{N} = \frac{N_s}{n}$$

- Standard error (standard deviation of the mean) is given by:

$$\sigma_{\bar{N}} = \frac{\sigma_{N_s}}{n} = \frac{\sqrt{N_s}}{n} = \frac{\sqrt{n\bar{N}}}{n} = \sqrt{\frac{\bar{N}}{n}}$$

- The standard deviation for a single measurement N_i is:

$$\sigma_{N_i} = \sqrt{N_i}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.1. Multiple independent counts

5.6.1.2. Mean value of multiple independent counts

- A typical count will not differ greatly from the mean

$$N_i \approx \bar{N}$$

- The mean value based on n independent counts will have an expected error that is smaller by a factor of \sqrt{n} compared with any single measurement on which the mean is based
- To improve the statistical precision of a given measurement by a factor of two, the counting time must, therefore, be increased four times

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.2. Standard deviation & relative standard deviation for counting rates

- If N counts are accumulated over time t , then the counting rate R is given by:

$$R = \frac{N}{t}$$

- Assuming the time t is measured with a very small uncertainty, so that t can be considered a constant

- Uncertainty of counting rate (an application of the propagation of errors multiplying by a constant) :

$$\sigma_R = \frac{\sigma_x}{t} = \frac{\sqrt{N}}{t} = \sqrt{\frac{R}{t}}$$

- Fractional standard deviation

$$\sigma_F = \frac{\sigma_x}{tR} = \frac{\sqrt{N}}{tR} = \sqrt{\frac{1}{tR}}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.2. Standard deviation & relative standard deviation for R

5.6.2.1. Example: comparison of error of R & counts accumulated

The activity of two samples is measured

Sample 1:

- The counts acquired:

$$N = 10\,000 \text{ counts}$$

- Standard deviation:

$$\sigma_N = 100 \text{ counts}$$

- Fractional standard deviation:

$$\sigma_F = 0.01 = 1\%$$

Sample 2:

- $R = 10\,000 \text{ cps}$

- Standard deviation:

$$\sigma_R = \sqrt{\frac{10000}{100}} = 10 \text{ cps}$$

- Fractional standard deviation:

$$\sigma_F = \sqrt{\frac{1}{10000 \times 100}} = 0.001 = 0.1\%$$

The counts acquired for samples 1 & 2 were numerically the same

The uncertainties were very different

Need to determine if value used is a single value or a value that has been obtained by calculation

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

□ Background counts

- Do not originate from the sample or target volume
- Unwanted counts such as scatter

□ Background counts during sample counting consist of

- Electronic noise
- Detection of cosmic rays
- Natural radioactivity in the detector
- Down scatter radioactivity from non-target radionuclides in the sample

□ Radiation from non-target tissue will also contribute to background during in vivo measurements

- e.g. measurement of thyroid iodine uptake or left ventricular ejection fraction



5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

❑ Scattered radiation from target as well as non-target tissue will influence quantification and will be included in the background

❑ If the background count is N_b , and the gross counts of the sample and background is N_g , then the net sample count N_s is:

$$N_s = N_g - N_b$$

❑ Standard deviation for N_s $\sigma(N_s) = \sqrt{N_g + N_b}$

❑ Fractional standard deviation for N_s

$$\sigma_F(N_s) = \frac{\sqrt{N_g + N_b}}{N_g - N_b}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

- If the background count rate is R_b , acquired in time t_b , and the gross count rate of the sample and background is R_g , acquired in time t_g , then the net sample counts R_s is:

$$R_s = R_g - R_b$$

- Standard deviation for R_s

$$\sigma(R_s) = \sqrt{\frac{R_g}{t_g} + \frac{R_b}{t_b}}$$

- Fractional standard deviation for N_s

$$\sigma_F(R_s) = \frac{\sqrt{\frac{R_g}{t_g} + \frac{R_b}{t_b}}}{R_g - R_b}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

□ If the same counting time t is used for both sample and background measurement:

- Standard deviation for R_s

$$\sigma(R_s) = \frac{\sqrt{R_g + R_b}}{\sqrt{t}}$$

- Fractional standard deviation for N_s

$$\sigma_F(R_s) = \frac{\sqrt{R_g + R_b}}{\sqrt{t} (R_g - R_b)}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

5.6.3.1. Example: error in net target counts following background correction

□ *Example:*

- Planar image of the liver is acquired for the detection of tumours
- ROI1 and ROI2 selected
 - equal sized regions
 - cover 2 potential tumours
- N_b selected over normal tissue of the same area as ROI1 & ROI2
- ROI1
 - $N_g = 484$ counts
 - $N_b = 441$ counts
- ROI2
 - $N_g = 484$ counts
 - $N_b = 169$ counts

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

5.6.3.1. Example: error in net target counts following background correction

□ *Example:*

- Difference

$$N_s = N_g - N_b = 484 - 441 = 43 \text{ counts}$$

- Error associated with the difference when $N_g \square N_b$:

$$\sigma(N_s) = \sqrt{484 + 441} = 30.4 \text{ counts}$$

$$\sigma_F(N_s) = \frac{\sqrt{484 + 441}}{484 - 441} = 0.7073$$

$$\sigma_P(N_s) = 70.7\%$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

5.6.3.1. Example: error in net target counts following background correction

□ *Example:*

$N_g \approx N_b$					$N_g \gg N_b$				
Source	Counts	σ counts	σ_F	σ_P (%)	Source	Counts	σ counts	σ_F	σ_P (%)
N_g	484	22.0	0.0455	4.5	N_g	484	22.0	0.0455	4.5
N_b	441	21.0	0.0476	4.8	N_b	169	13.0	0.0769	7.7
N_s	43	30.4	0.7073	70.7	N_s	315	25.6	0.0811	8.1
$3\sigma(N_s)$	91	Counts	Not significant		$3\sigma(N_s)$	77	Counts	Significant	

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

5.6.3.1. Example: error in net target counts following background correction

- Conclusion:
 - σ_F and percentage σ_P significantly increase when the background increases relative to the net counts

- It is important to acquire as many counts as possible to decrease the uncertainty in detection of target volume radioactivity

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

5.6.3.1. Example: error in net target counts following background correction

- ❑ Planar image of the liver is acquired for the detection of tumours
- ❑ ROI1 and ROI2 selected
 - Equal sized regions
 - Cover 2 potential tumours
- ❑ The acquisition time of the image was 2 min
- ❑ R_b selected over normal tissue of the same area as ROI1 & ROI2
 - ROI1
 - $R_g = 484$ cpm
 - $R_b = 441$ cpm
 - ROI2
 - $R_g = 484$ cpm
 - $R_b = 169$ cpm

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

5.6.3.1. Example: error in net target counts following background correction

□ **Example:**

- Difference

$$R_s = R_g - R_b = 484 - 441 = 43 \text{ cpm}$$

- Error associated with the difference when $N_g \square N_b$:

$$\sigma(R_s) = \sqrt{\frac{484}{2} + \frac{441}{2}} = 21.5 \text{ cpm}$$

$$\sigma_F(R_s) = \frac{\sqrt{\frac{484}{2} + \frac{441}{2}}}{484 - 441} = 0.5001$$

$$\sigma_P(R_s) = 50.0\%$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.3. Effects of background counts

5.6.3.1. Example: error in net target counts following background correction

□ **Example:**

$N_g \approx N_b$					$N_g \gg N_b$				
Source	Counts	σ counts	σ_F	σ_P (%)	Source	Counts	σ counts	σ_F	σ_P (%)
N_g	484	22.0	0.0455	4.5	N_g	484	22.0	0.0455	4.5
N_b	441	21.0	0.0476	4.8	N_b	169	13.0	0.0769	7.7
N_s	43	30.4	0.7073	70.7	N_s	315	25.6	0.0811	8.1
$3\sigma(N_s)$	91	Counts	Not significant		$3\sigma(N_s)$	77	Counts	Significant	

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.5. Minimum detectable counts, count rate and activity

- If the difference of 2 measurements is larger than 3σ , the difference is considered significant.
- Therefore, the minimum net counts N_m that can be detected with 0.3% confidence is given by:

$$N_m = N_1 - N_2 = 3\sigma(N_1 - N_2)$$

or

$$N_m = N_g - N_b = 3\sigma(N_g - N_b) = 3\sqrt{N_g + N_b}$$

- N_g will give the minimum detectable gross counts N_m :

$$N_m = \frac{(2N_b + 9) + \sqrt{72N_b + 81}}{2}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.5. Minimum detectable counts, count rate and activity

- An approximation can be used by assuming that $N_g \approx N_b$ and:

$$N_g \approx N_b + 3\sqrt{2N_b}$$

- The minimum detectable activity A_m can be calculated

$$A_m = \frac{N_m}{tS}$$

where S is the sensitivity of the detection system (count rate per becquerel),

t is the time that the background was counted

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.5. Minimum detectable counts, count rate and activity

5.6.5.1. Example: calculation of minimum activity that can be detected

□ A detector is to be used to detect ^{131}I in thyroid of radiation workers

- $N_b = 441$ counts
- $t_b = 5$ min
- Acquisition time = 5 min
- Counter sensitivity = $0.1 \text{ counts} \cdot \text{s}^{-1} \cdot \text{Bq}^{-1}$
- What is the minimum activity that can be detected?

$$N_g = \frac{(2N_b + 9) + \sqrt{72N_b + 81}}{2} = \frac{(2 \times 441 + 9) + \sqrt{72 \times 441 + 81}}{2} = 535$$

$$N_g - N_b = 94 \text{ counts}$$



$$A_m = \frac{(535 - 441)}{5 \times 60 \times 0.1} = 3.124 \text{ Bq}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.5. Minimum detectable counts, count rate and activity

5.6.5.1. Example: calculation of minimum activity that can be detected

□ The minimum detectable net count rate:

- $N_b = 441$ counts
- $t_b = 5$ min
- Acquisition time = 5 min
- Counter sensitivity = $0.1 \text{ counts} \cdot \text{s}^{-1} \cdot \text{Bq}^{-1}$
- What is the minimum activity that can be detected?

$$R_m = R_g - R_b > 3\sigma(R_g - R_b)$$

$$R_g = \frac{\left(2R_b + \frac{9}{t_g}\right) + \sqrt{\frac{36R_b}{t_g} + \frac{81}{t_g^2} + \frac{36R_b}{t_b}}}{2}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.5. Minimum detectable counts, count rate and activity

5.6.5.2. Example 2: calculation of minimum activity that can be detected

□ Assuming that $R_g \sim R_b$:

$$R_g \approx R_b + 3 \sqrt{\frac{R_b}{t_g} + \frac{R_b}{t_b}}$$

□ The minimum detectable activity A_m can be calculated:

$$A_m = \frac{R_m}{S}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.5. Minimum detectable counts, count rate and activity

5.6.5.2. Example 2: calculation of minimum activity that can be detected

□ The minimum detectable net count rate:

- $N_b = 441$ cpm
- $t_b = 5$ min
- Acquisition time = 1 min
- Counter sensitivity = $0.1 \text{ counts} \cdot \text{s}^{-1} \cdot \text{Bq}^{-1}$
- What is the minimum activity that can be detected?

$$R_g = \frac{\left(2 \times 441 + \frac{9}{1}\right) + \sqrt{36 \times \frac{441}{1} + \frac{81}{1^2} + 36 \times \frac{441}{5}}}{2} = 515 \text{ cpm}$$

$$R_g - R_b = 74 \text{ cpm}$$

$$3\sigma(R_g - R_b) = 74 \text{ cpm}$$

$$A_m = \frac{(515 - 441)}{0.1 \times 60} = 12.28 \text{ Bq}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.6. Comparing counting systems

□ Important points:

- Large number of counts have smaller σ_F
- σ_F increases rapidly as N_b increases
- High sensitivity counting system with low background
- But as sensitivity increases the system will also be more sensitive to background

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.6. Comparing counting systems

□ Analysis of sensitivity/background trade-off

- Compare results from systems 1 and 2
- t is the same for N_g and N_b

$$\sigma_{F1}(N_{S1}) = \frac{\sqrt{N_{g1} + N_{b1}}}{N_{g1} - N_{b1}} \qquad \sigma_{F2}(N_{S2}) = \frac{\sqrt{N_{g2} + N_{b2}}}{N_{g2} - N_{b2}}$$

- σ_F for net sample counts obtained with the 2 systems are:

$$\frac{\sigma_{F1}(N_{S1})}{\sigma_{F2}(N_{S2})} = \frac{\frac{\sqrt{N_{g1} + N_{b1}}}{N_{g1} - N_{b1}}}{\frac{\sqrt{N_{g2} + N_{b2}}}{N_{g2} - N_{b2}}}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.6. Comparing counting systems

- ❑ System 1 is statistically the preferred system if:

$$\frac{\sigma_{F1}(N_{S1})}{\sigma_{F2}(N_{S2})} < 1$$

- ❑ System 2 is statistically the preferred system if:

$$\frac{\sigma_{F1}(N_{S1})}{\sigma_{F2}(N_{S2})} > 1$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.6. Comparing counting systems

- Systems can be compared using R_S & σ_F for the count rate using the ratio of the σ_F 's:

$$\frac{\sigma_{F1}(R_{S1})}{\sigma_{F2}(R_{S2})} = \frac{\sqrt{\frac{R_{g1}}{t_{g1}} + \frac{R_{b1}}{t_{b1}}}}{\frac{R_{g1} - R_{b1}}{\sqrt{\frac{R_{g2}}{t_{g2}} + \frac{R_{b2}}{t_{b2}}}}}$$

- Use to compare different counting times in the same system for measuring fixed geometry samples

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.6. Comparing counting systems

- ❑ To obtain the best energy window selection in a system, or to compare 2 systems, the same t should be used:

$$\frac{\sigma_{F1}(R_{S1})}{\sigma_{F2}(R_{S2})} = \frac{\frac{\sqrt{R_{g1} + R_{b1}}}{R_{g1} - R_{b1}}}{\frac{\sqrt{R_{g2} + R_{b2}}}{R_{g2} - R_{b2}}}$$

- ❑ Can also be used in planar imaging
- ❑ Different collimators can be evaluated by comparing counts from a target region to a non-target or background region
- ❑ Spatial resolution is also important and must be considered

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.7. Estimating required counting times

□ Suppose

- Want to determine R_s to within a certain $\sigma_F(R_s)$
- Suppose R_{ga} & R_{ba} are known from preliminary measurements
- t is the same for the sample & background
- The time required to achieve the desired statistical reliability

$$t = \frac{R_{ga} + R_{ba}}{\left[\sigma_F^2(R_s) \right] (R_{ga} - R_{ba})^2}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.7. Estimating required counting

5.6.7.1. Example: calculation of required counting time

- Determine the counting time for a thyroid uptake study using a collimated detector
 - Preliminary target measurement of $R_{ga} = 900$ cpm
 - Preliminary background $R_{ba} = 100$ cpm
 - What counting time is required to determine the net count rate to within 5%?
 - $t = 0.625$ min for both target & background
 - Total time = 1.25 min

$$R_{sa} = 900 - 100 = 800 \text{ cpm}$$

$$t = \frac{900 + 100}{(0.05)^2 \times (900 - 100)^2} = \frac{1000}{(0.05)^2 \times (800)^2} = 0.625 \text{ min}$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.8. Calculating uncertainties in the measurement of plasma volume in patients

□ Plasma volume (PV) and its uncertainty is to be measured using the dilution principle

- Procedure

- Prepare plasma sample of known volume
- Prepare standard sample with same activity & volume
- Dilute standard sample and then count
- Obtain blood sample 10 min after injection
- Separate plasma from the blood
- Count blood sample
- Calculate PV by:

$$PV = \frac{R_s}{R_p} VD$$

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.8. Calculating uncertainties in the measurement of plasma volume in patients

The uncertainties are calculated step by step by applying the propagation of errors:

□ Using the values:

- Counting time $t = 10$ min
- $V \pm \sigma_p(V) = 5 \pm 3\%$ mL
- $D \pm \sigma_p(D) = 200 \pm 3\%$
- $R_{s+b} = 3200$ cpm
- $R_{p+b} = 1200$ cpm
- $R_b = 200$ cpm

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.8. Calculating uncertainties in the measurement of plasma volume in patients

□ Using the expressions:

- Standard sample net count rate per mL
 - $R_s = R_{s+b} - R_b$;
- R_b = background count rate
- R_{s+b} = standard sample gross count rate per mL
- Plasma sample net count rate per mL
 - $R_p = R_{p+b} - R_b$;
- R_{p+b} = plasma sample gross count rate per mL
- V = volume of standard sample in mL
- $\sigma_p(V)$ = percentage uncertainty in V
- D = dilution of standard sample
- $\sigma_p(D)$ = percentage uncertainty in D

5.6. APPLICATIONS OF STATISTICAL ANALYSIS

5.6.8. Calculating uncertainties in the measurement of plasma volume in patients

Values			Uncertainty in values			
Symbol	Value	Unit	Symbol	Calculation	σ	$\sigma_F(\%)$
t	10	min				
R_{s+b}	3200	cpm	$\sigma(R_{s+b})$	$\sqrt{\frac{R_{s+b}}{t}}$	17.89	0.559
R_b	200	cpm	$\sigma(R_b)$	$\sqrt{\frac{R_b}{t}}$	4.472	2.236
R_s	3000	cpm	$\sigma(R_s)$	$\sqrt{(\sigma(R_{s+b}))^2 + (\sigma(R_b))^2}$	18.44	0.615
R_{p+b}	1200	cpm	$\sigma(R_{p+b})$	$\sqrt{\frac{R_{p+b}}{t}}$	10.95	0.913
R_p	1000	cpm	$\sigma(R_p)$	$\sqrt{(\sigma(R_{p+b}))^2 + (\sigma(R_b))^2}$	11.83	1.183
$\frac{R_s}{R_p}$	3		$\sigma(R_s/R_p)$	$\frac{R_s}{R_p} \sqrt{\left(\frac{\sigma(R_s)}{R_s}\right)^2 + \left(\frac{\sigma(R_p)}{R_p}\right)^2}$	0.040	1.333
V	5	mL	$\sigma(V)$		0.150	3.000
$\frac{R_s}{R_p} V$	15	mL	$\sigma\left(\frac{R_s}{R_p} V\right)$	$\left(\frac{R_s}{R_p} V\right) \sqrt{\left(\frac{\sigma(R_s/R_p)}{R_s/R_p}\right)^2 + \left(\frac{\sigma(V)}{V}\right)^2}$	0.492	3.283
D	200		$\sigma(D)$		6.000	3.000
$PV = \frac{R_s}{R_p} VD$	3000	mL	$\sigma(PV)$	$PV \sqrt{\left(\frac{\sigma\left(\left(\frac{R_s}{R_p} V\right)\right)}{\left(\frac{R_s}{R_p} V\right)}\right)^2 + \left(\frac{\sigma(D)}{D}\right)^2}$	133	4.447

5.7. APPLICATION OF STATISTICAL ANALYSIS: DETECTOR PERFORMANCE

5.7.1. Energy resolution of scintillation detect

- ❑ Poisson statistics affect other aspects of the detection of radiation:
 - Determines detector energy resolution
 - Determines uncertainty of energy measurement of detected photon
- ❑ Energy resolution / uncertainty determined by:
 - Detector type
 - ➡ e.g. solid state detector energy resolution significantly better than of scintillation detector
 - Photon energy
- ❑ Energy resolution for detector / radionuclide does not change between samples
 - Different from statistics uncertainty determined by number of counts accumulated

5.7. APPLICATION OF STATISTICAL ANALYSIS: DETECTOR PERFORMANCE

5.7.1. Energy resolution of scintillation detect

- ❑ Even for the same sample and same detector system, the uncertainty can change if measurements are repeated following the decay of the nuclide
- ❑ Another important consequence of statistics is that in scintillation cameras the location of the position of incoming photons is based on the pulses detected by the detectors
- ❑ Therefore, the statistics of the detector system limits the spatial resolution that can be achieved with an imaging device. A clear understanding of the statistics associated with the detector when detecting a photon is, therefore, important

5.7. APPLICATION OF STATISTICAL ANALYSIS: DETECTOR PERFORMANCE

5.7.1. Energy resolution of scintillation detect

□ The operation of scintillation detectors can be considered a three stage process:

- x = number of light photons produced in scintillator by detected gamma rays
- p = fraction of light photons that will eject electrons from the photocathode of the photomultiplier tube (PMT)
- M = multiplication electrons at successive dynodes before being collected at the anode

5.7. APPLICATION OF STATISTICAL ANALYSIS: DETECTOR PERFORMANCE

5.7.1. Energy resolution of scintillation detect

- Average number of electrons produced at the anode:

$$N_e = xpM$$

- Fractional variance in the electron number N for a 3 stage cascade process:

$$\sigma_F^2(N_e) = \sigma_F^2(x) + \sigma_F^2(xp) + \sigma_F^2(xpM)$$

- For dynodes with identical M

$$\sigma_F^2(M) = \frac{1}{M-1}$$

- Assuming the production of light photons follows a Poisson distribution:

$$\sigma_F^2(x) = \frac{1}{x} \quad \sigma_F^2(p) = \frac{1-p}{p}$$

5.7. APPLICATION OF STATISTICAL ANALYSIS: DETECTOR PERFORMANCE

5.7.1. Energy resolution of scintillation detect

□ Fractional variance in the electron number:

$$\sigma_F^2(N_e) = \frac{1}{x} + \frac{1}{x} \frac{1-p}{p} + \frac{1}{xp} \frac{1}{(M-1)}$$

□ Fractional energy resolution (R_E) of detectors:

$$R_E = \frac{\text{FWHM}}{E} = \frac{2.35\sigma(E)}{E} = \frac{2.35\sigma(N_e)}{N_e}$$

$$R_E = 2.355 \sqrt{\frac{1}{x \left(1 + \frac{1-p}{p} + \frac{1}{p(M-1)} \right)}}$$

5.7. APPLICATION OF STATISTICAL ANALYSIS: DETECTOR PERFORMANCE

5.7.2. Intervals between successive events

- ❑ Calculation and measurement of the paralyzable dead time of counting systems involves time intervals separating random events
 - r = average rate at which events are occurring
 - $r dt$ = differential probability for event taking place in the differential time increment dt

- ❑ For a radiation detector with unity efficiency, the time interval for counting a single radionuclide is given by:

$$r = \left| \frac{dN}{dt} \right| = \lambda N$$

5.7. APPLICATION OF STATISTICAL ANALYSIS: DETECTOR PERFORMANCE

5.7.2. Intervals between successive events

- To derive the distribution function to describe the time interval between adjacent random events
 - Assume an event has occurred at $t = 0$
 - Two independent processes must take place:
 - No events may occur within the time interval from 0 to t
 - An event must take place in the next differential time increment dt
 - Overall probability given by :

$$P_1(t) dt = P(0) \times r dt$$

- $P_1(t) dt$ = Probability of next event taking place dt after delay of t
- $P(0)$ = Probability of number of events during time from 0 to t
- $r dt$ = Probability of event during dt

5.7. APPLICATION OF STATISTICAL ANALYSIS: DETECTOR PERFORMANCE

5.7.2. Intervals between successive events

- The possibility that no events will be recorded over an interval of t

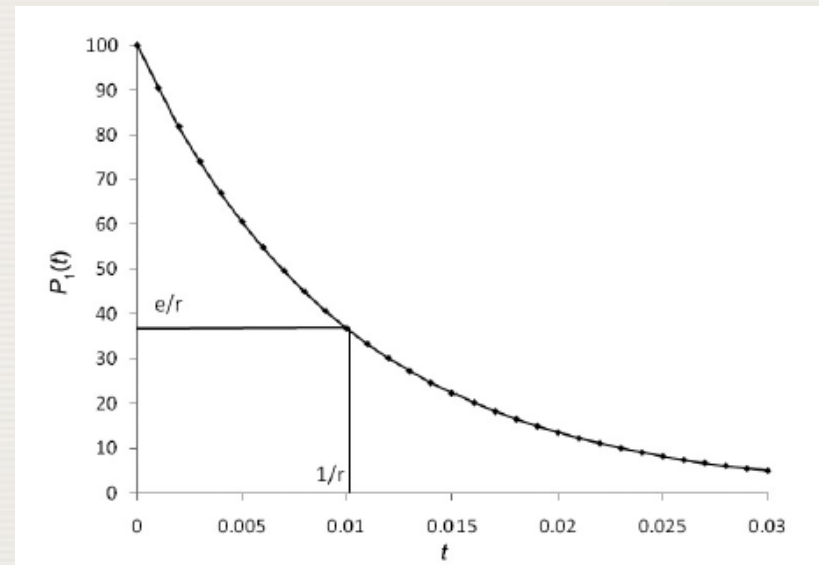
$$P(0) = \frac{(rt)^0 e^{-rt}}{0!} = e^{-rt}$$

- rt = average number of recorded events
- Substituting

$$P_1(t)dt = re^{-rt} dt$$

- Most probable distribution is zero
- Average interval length:

$$\bar{t} = \frac{\int_0^{\infty} tP_1(t) dt}{\int_0^{\infty} P_1(t) dt} = \frac{\int_0^{\infty} te^{-rt} dt}{\int_0^{\infty} e^{-rt} dt} = \frac{1}{r}$$



5.7. APPLICATION OF STATISTICAL ANALYSIS: DETECTOR PERFORMANCE

5.7.3. Paralyzable dead time

□ Paralyzable dead time model

- τ = fixed dead time following each event during detector live period
- Events occurring during τ
 - Not recorded
 - Create another fixed dead time τ on the lost event
 - m = recorded rate of events
 - identical to rate of occurrences of time intervals $> \tau$ between trues
- The probability of intervals $> \tau$:

$$P_2(t) dt = \int_{\tau}^{\infty} P_1(t) dt = e^{-r\tau}$$

- Rate of occurrence m of such intervals :

$$m = re^{-r\tau}$$

- must be solved iteratively to calculate r from measurements of m and τ

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